

Casimir force induced by an impenetrable flux tube of finite radius

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Abstract

A perfectly reflecting (Dirichlet) boundary condition at the edge of an impenetrable magnetic-flux-carrying tube of nonzero transverse size is imposed on the charged massive scalar matter field which is quantized outside the tube. We show that the vacuum polarization effects give rise to a macroscopic force acting at the increase of the tube radius under the steady magnetic flux.

1 Introduction

Polarization of the vacuum of quantized matter fields under the influence of boundary conditions was studied intensively over more than six decades since Casimir [1] predicted a force between grounded metal plates: the prediction was that the induced vacuum energy in bounded spaces gave rise to a macroscopic force between bounding surfaces, see reviews in Refs. [2] and [3]. The Casimir force between grounded metal plates has now been measured quite accurately and agrees with the theoretical predictions, see, e.g. Refs. [4] and [5], as well as other publications cited in Refs. [2] and [3].

In the present paper we consider the vacuum energy which is induced by boundary conditions in space that is not bounded but, instead, is not simply connected, being an exterior to a straight infinitely long tube. This setup is inspired by the famous Aharonov-Bohm effect [6], and we are interested in polarization of the vacuum which is due to imposing a boundary condition at the edge of the tube carrying magnetic flux lines inside itself; this may be denoted as the Casimir-Aharonov-Bohm effect (see also [7]).

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The temporal component of the energy-momentum tensor for the quantized charged scalar field $\Psi(x)$ in flat space-time is given by expression

$$T_{00}(x) = \frac{1}{2} [\partial_0 \Psi^\dagger, \partial_0 \Psi]_+ - \frac{1}{4} [\partial_0^2 \Psi^\dagger, \Psi]_+ - \frac{1}{4} [\Psi^\dagger, \partial_0^2 \Psi]_+ - \left(\xi - \frac{1}{4} \right) \nabla^2 [\Psi^\dagger, \Psi]_+, \quad (1)$$

where ∇ is the covariant spatial derivative involving both affine and bundle connections and the field operator in the case of a static background takes form

$$\Psi(x^0, \mathbf{x}) = \sum_{\lambda} \frac{1}{\sqrt{2E_{\lambda}}} \left[e^{-iE_{\lambda}x^0} \psi_{\lambda}(\mathbf{x}) a_{\lambda} + e^{iE_{\lambda}x^0} \psi_{-\lambda}(\mathbf{x}) b_{\lambda}^{\dagger} \right]; \quad (2)$$

a_{λ}^{\dagger} and a_{λ} (b_{λ}^{\dagger} and b_{λ}) are the scalar particle (antiparticle) creation and destruction operators satisfying commutation relations; wave functions $\psi_{\lambda}(\mathbf{x})$ form a complete set of solutions to the stationary Klein-Gordon equation

$$(-\nabla^2 + m^2) \psi_{\lambda}(\mathbf{x}) = E_{\lambda}^2 \psi_{\lambda}(\mathbf{x}), \quad (3)$$

m is the mass of the scalar particle; λ is the set of parameters (quantum numbers) specifying the state; $E_{\lambda} = E_{-\lambda} > 0$ is the energy of the state; symbol \sum_{λ} denotes summation over discrete and integration (with a certain measure) over continuous values of λ .

As is known for a long time [8, 9, 10], the energy-momentum tensor depends on parameter ξ which couples Ψ to the scalar curvature of space-time even in the case of the vanishing curvature, see (1); conformal invariance is achieved in the limit of vanishing mass ($m = 0$) at $\xi = (d-1)(4d)^{-1}$, where d is the spatial dimension. Consequently, the density of the induced vacuum energy which is given formally by expression

$$\varepsilon = \langle \text{vac} | T_{00}(x) | \text{vac} \rangle = \sum_{\lambda} E_{\lambda} \psi_{\lambda}^*(\mathbf{x}) \psi_{\lambda}(\mathbf{x}) - (\xi - 1/4) \nabla^2 \sum_{\lambda} E_{\lambda}^{-1} \psi_{\lambda}^*(\mathbf{x}) \psi_{\lambda}(\mathbf{x}) \quad (4)$$

depends on ξ as well. This poses a question: whether physically measurable effects (i.e. the Casimir force) can be dependent on ξ ?

In the present paper we are considering a static background in the form of the cylindrically symmetric magnetic flux tube of finite transverse size, hence the covariant derivative is $\nabla = \partial - ie\mathbf{V}$ with the vector potential possessing only one nonvanishing component given by

$$V_{\varphi} = \Phi/2\pi \quad (5)$$

outside the tube; here Φ is the value of the magnetic flux and φ is the angle in polar (r, φ) coordinates on a plane which is transverse to the tube. The Dirichlet boundary condition at the edge of the tube ($r = r_0$) is imposed on the scalar field:

$$\psi_{\lambda}|_{r=r_0} = 0, \quad (6)$$

i.e. quantum matter is assumed to be perfectly reflected from the thence impenetrable flux tube.

As we shall see, the vacuum energy which is induced outside the flux tube gives rise to a macroscopic force acting at the increase of the tube radius under the steady magnetic flux. Although the induced vacuum energy density is ξ -dependent, the Casimir force will be shown to be independent of ξ .

2 Vacuum energy density

The solution to (3) outside the magnetic flux tube can be obtained in terms of the cylindrical functions. The formal expression (4) for the vacuum energy density has to be renormalized by subtracting the contribution corresponding to the zero flux. Restricting ourselves to a plane which is orthogonal to the tube, we obtain (for details see [11]):

$$\varepsilon_{ren} = \frac{1}{2\pi} \left\{ \int_0^\infty dk k (k^2 + m^2)^{1/2} [S(kr, kr_0) - S(kr, kr_0)|_{\Phi=0}] - (\xi - 1/4)\Delta \int_0^\infty dk k (k^2 + m^2)^{-1/2} [S(kr, kr_0) - S(kr, kr_0)|_{\Phi=0}] \right\}, \quad (7)$$

where

$$S(kr, kr_0) = \sum_{n \in \mathbb{Z}} \frac{[Y_{|n-e\Phi/2\pi|}(kr_0)J_{|n-e\Phi/2\pi|}(kr) - J_{|n-e\Phi/2\pi|}(kr_0)Y_{|n-e\Phi/2\pi|}(kr)]^2}{Y_{|n-e\Phi/2\pi|}^2(kr_0) + J_{|n-e\Phi/2\pi|}^2(kr_0)}, \quad (8)$$

\mathbb{Z} is the set of integer numbers, $J_\mu(u)$ and $Y_\mu(u)$ are the Bessel functions of order μ of the first and second kinds, and $\Delta = \partial_r^2 + r^{-1}\partial_r$ is the radial part of the Laplacian operator on the plane.

Owing to the infinite range of summation, the last expression is periodic in flux Φ with a period equal to $2\pi e^{-1}$, i.e. the London flux quantum (we use units $c = \hbar = 1$). Our further analysis concerns the case of $\Phi = (2n+1)\pi e^{-1}$ when each of the integrals in (7) is the most distinct from zero. Introducing function [12]

$$G(kr, kr_0) = S(kr, kr_0)|_{\Phi=\pi e^{-1}} - S(kr, kr_0)|_{\Phi=0}, \quad (9)$$

we rewrite (7) in the dimensionless form

$$r^3 \varepsilon_{ren} = \alpha_+(mr_0, mr) - (\xi - 1/4)r^3 \Delta \frac{\alpha_-(mr_0, mr)}{r}, \quad (10)$$

where

$$\alpha_\pm(mr_0, mr) = \frac{1}{2\pi} \int_0^\infty dz z \left[z^2 + \left(\frac{mr_0}{\lambda} \right)^2 \right]^{\pm 1/2} G(z, \lambda z), \quad (11)$$

and $\lambda = r_0/r$ ($\lambda \in [0, 1]$).

We follow the technique of numerical calculations developed in [11, 12] with some modifications, notably we perform direct integration over consecutive periods of the $G(z, \lambda z)$ function using the Euler-Maclaurin integration formula [13]. This results in a sufficient decrease of the computation time.

Thereafter we calculate α_+ and α_- functions for the case of $mr_0 = 10^{-3}$ at a set of different distances from the axis of the tube. This allows us to obtain coefficients of the interpolation function which is approximated in the form

$$\alpha_\pm(x_0, x) = [\pm e^{-2x} x^{1 \mp 1/2}] \left[\left(\frac{x - x_0}{x} \right)^2 \frac{P_3^\pm(x - x_0)}{x^3} \right] \frac{Q_3^\pm(x^2)}{x^6}, \quad x > x_0, \quad (12)$$

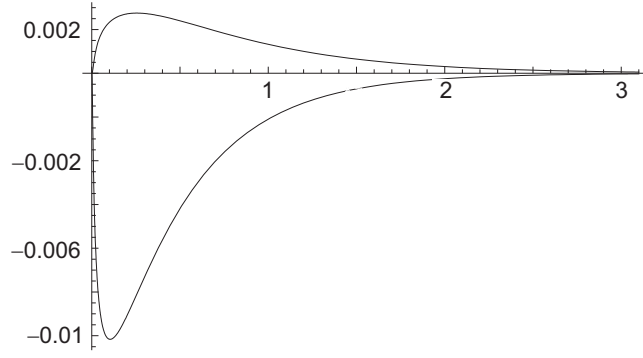


Figure 1: Behavior of the $\alpha_+(x_0, x)$ (positive) and the $\alpha_-(x_0, x)$ (negative) functions for the case of $x_0 = 10^{-3}$. The variable x ($x > x_0$) is along the abscissa axis.

where $x = mr$, $x_0 = mr_0$ and $P_n^\pm(y)$, $Q_n^\pm(y)$ — are polynomials in y of the n -th order with the x_0 -dependent coefficients. First factor in square bracket in (12) describes the large distance behavior in the case of the zero-radius tube (singular thread), second factor in square bracket is an asymptotics at small distances from the edge of the tube, and the last factor in square bracket is the intermediate part. Since the flux tube is impenetrable, the α_\pm functions vanish at $x \leq x_0$. Behavior of the dimensionless α_\pm functions is presented on Fig.1.

For the α_+ function we estimate the relative error of the obtained result as 0.1%. It should be noted that nearly 95 % of the integral value is obtained by direct calculation and only nearly 5% is the contribution from the interpolation. The integration in the case of the α_- function is performed more quickly and with a higher accuracy, as compared to the case of the α_+ function, because the former tends to zero more rapidly at large distances. In this case the contribution from the interpolation can be estimated as $10^{-3}\%$ from the total value.

We define function [11]

$$\tilde{\alpha}_-(x_0, x) = r^3 \Delta \frac{\alpha_-(x_0, x)}{r} = \alpha_-(x_0, x) - x \frac{\partial \alpha_-(x_0, x)}{\partial x} + x^2 \frac{\partial^2 \alpha_-(x_0, x)}{\partial x^2} \quad (13)$$

and present its behavior on Fig.2.

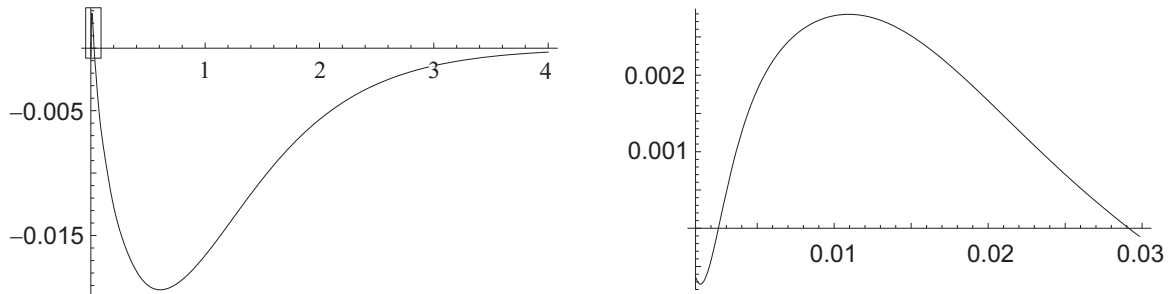


Figure 2: Behavior of the $\tilde{\alpha}_-(x_0, x)$ function for the case of $x_0 = 10^{-3}$. The region in a rectangle on the left figure is seen in the scaled-up form on the right figure. The variable x ($x > x_0$) is along the abscissa axis.

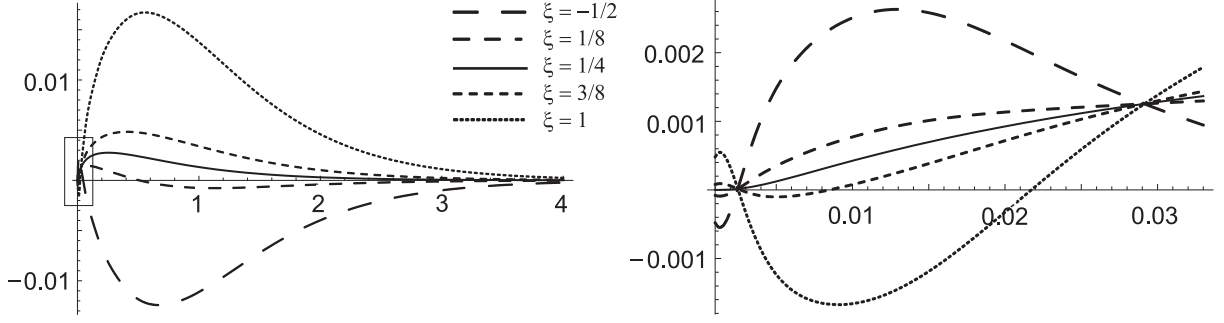


Figure 3: The dimensionless vacuum energy density $r^3 \varepsilon_{ren}(x_0, x)$ at different values of the coupling to the space-time curvature scalar for the case of $x_0 = 10^{-3}$. The region in a rectangle on the left figure is seen in the scaled-up form on the right figure. The variable x ($x > x_0$) is along the abscissa axis.

We construct the dimensionless vacuum energy density at different values of the coupling to the space-time curvature scalar (ξ):

$$r^3 \varepsilon_{ren} = \alpha_+(x_0, x) - (\xi - 1/4) \tilde{\alpha}_-(x_0, x) \quad (14)$$

and present its behavior in the case of $mr_0 = 10^{-3}$ on Fig.3.

The behavior of the induced vacuum energy density as the radius of the tube tends to zero is of primary interest. To do a numerical calculation at $x_0 < 10^{-3}$ needs a long computational time and is a rather complicated task. Nevertheless, we can make some general conclusions regarding the case of small x_0 .

It seems plausible that this case with the decrease of the tube radius becomes more similar to the case of the tube of zero radius (singular thread) see, e.g., Fig.4. However there are some peculiarities in the behavior in the vicinity of the tube. To discuss them, let us first recall the exact expressions corresponding to the case of the singular magnetic thread (see [14]):

$$\alpha_+(0, x) = \frac{x^3}{3\pi^2} \left\{ \frac{\pi}{2} - 2xK_0(2x) - K_1(2x) + \frac{K_2(2x)}{2x} - \right. \quad (15)$$

$$\left. - \pi x [K_0(2x)L_1(2x) + K_1(2x)L_0(2x)] \right\}, \quad (16)$$

$$\alpha_-(0, x) = \frac{x}{\pi^2} \left\{ \frac{\pi}{2} - 2xK_0(2x) - K_1(2x) - \pi x [K_0(2x)L_1(2x) + K_1(2x)L_0(2x)] \right\}, \quad (17)$$

$$\tilde{\alpha}_-(0, x) = -\frac{x}{\pi^2} [2xK_0(2x) + K_1(2x)], \quad (18)$$

where $K_\nu(u)$ and $L_\nu(u)$ are the Macdonald and the modified Struve functions of order ν .

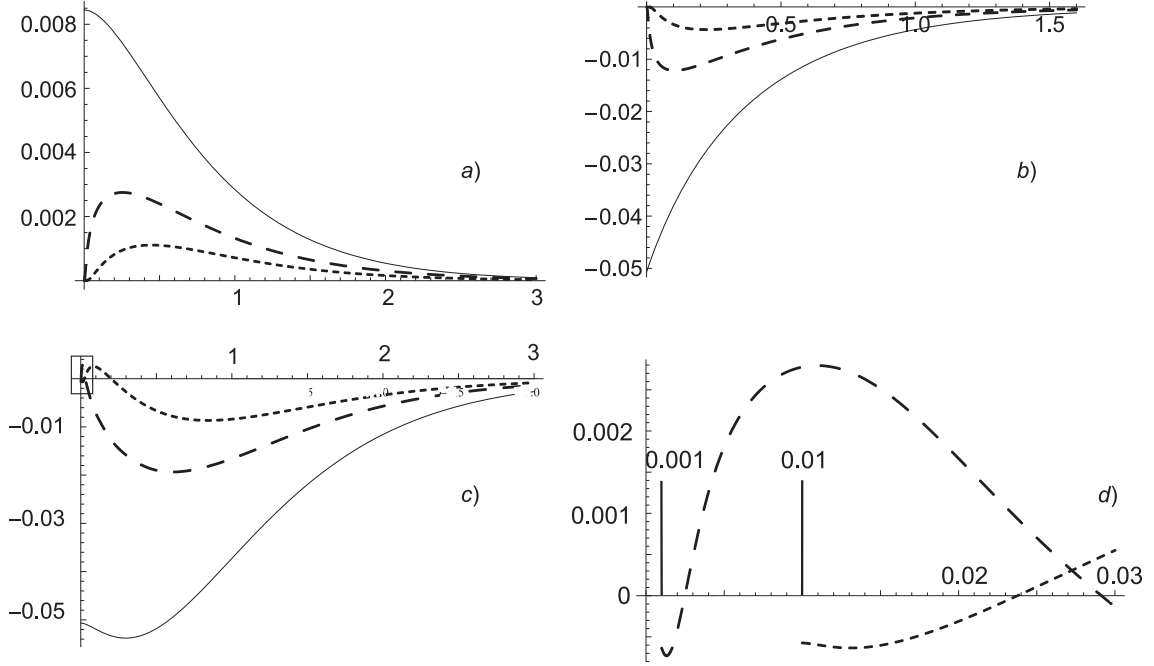


Figure 4: The constituents of the dimensionless vacuum energy density: *a*) α_+ , *b*) α_- , *c*) $\tilde{\alpha}_-$ for the case of $x_0 = 10^{-2}$ (dotted line), 10^{-3} (dashed line). The region in a rectangle on the *c*-figure is seen in the scaled-up form on the *d*-figure. The behavior of the corresponding functions for the case of a singular magnetic thread is presented by solid line. The variable x ($x > 0$) is along the abscissa axis.

Consequently, in the vicinity of a thread one gets

$$\alpha_+(0, x) = \frac{1 - 3x^2}{12\pi^2}, \quad x \ll 1 \quad (19)$$

$$\alpha_-(0, x) = -\frac{1 - \pi x + (3 - 2\gamma - 2 \ln x)x^2}{2\pi^2} + O(x^3), \quad x \ll 1, \quad (20)$$

$$\tilde{\alpha}_-(0, x) = -\frac{1}{2\pi^2} + \frac{1 + 2\gamma + 2 \ln x}{2\pi^2} x^2 + O(x^3), \quad x \ll 1, \quad (21)$$

where γ is the Euler constant. Using the latter relations, we get asymptotics of the renormalized vacuum energy density at small distances from the singular magnetic thread

$$r^3 \varepsilon_{ren}^{sing} = \frac{1}{12\pi^2} - \frac{x^2}{4\pi^2} - \left(\xi - \frac{1}{4} \right) \left(-\frac{1}{2\pi^2} + \frac{1 + 2\gamma + 2 \ln x}{2\pi^2} x^2 \right) + O(x^3), \quad x \ll 1. \quad (22)$$

In contrast to (19) and (20), the $\alpha_{\pm}(x_0, x)$ functions in the case of nonzero radius are vanishing quadratically in the vicinity of the tube, see [11],

$$\alpha_{\pm}(x_0, x)|_{x \rightarrow x_0} \sim O[(x - x_0)^2]. \quad (23)$$

To be more precise, we assume the asymptotics in the form, cf. (12),

$$\alpha_{\pm}(x_0, x) = \pm \frac{(x - x_0)^2}{x^2} f_{\pm}(x_0, x), \quad (24)$$

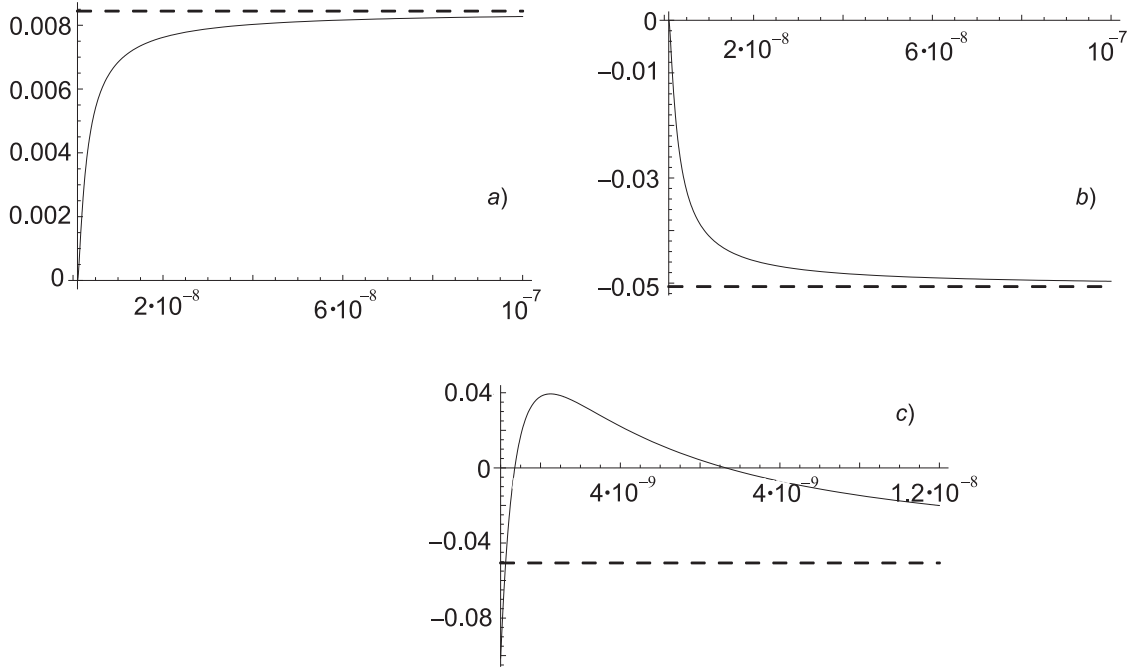


Figure 5: The expected behavior of the constituents of the dimensionless vacuum energy density at small distances from the tube: a) α_+ , b) α_- , c) $\tilde{\alpha}_-$ for the case of $x_0 = 10^{-9}$ (solid line). The behavior of the corresponding functions for the case of a singular magnetic thread is presented by dashed line. The variable x ($x > x_0$) is along the abscissa axis.

then one gets

$$\tilde{\alpha}_-(x_0, x) = -(x - x_0)^2 \frac{\partial^2}{\partial x^2} f_-(x_0, x) + \left(1 - 6\frac{x_0}{x} + 5\frac{x_0^2}{x^2}\right) x \frac{\partial}{\partial x} f_-(x_0, x) - \left(1 - 8\frac{x_0}{x} + 9\frac{x_0^2}{x^2}\right) f_-(x_0, x), \quad (25)$$

with $\tilde{\alpha}_-(x_0, x_0) = -2f_-(x_0, x_0)$.

The $f_{\pm}(x_0, x)$ functions are adjusted as

$$f_+(0, x) = \frac{1 - 3x^2}{12\pi^2}, \quad x \ll 1, \quad (26)$$

$$f_-(0, x) = \frac{1 - \pi x + (3 - 2\gamma - 2 \ln x)x^2}{2\pi^2}, \quad x \ll 1; \quad (27)$$

consequently, one gets

$$\tilde{\alpha}_-(x_0, x) \Big|_{\substack{x_0 \rightarrow 0 \\ x \rightarrow x_0}} = -\frac{1}{2\pi^2} + \frac{1 + 2\gamma + 2 \ln x}{2\pi^2} x^2 + \frac{4 - \pi x}{\pi^2 x} x_0 + \frac{-9 + 4\pi x - 7x^2 + 2\gamma x^2 + 2x^2 \ln x}{2\pi^2 x^2} x_0^2. \quad (28)$$

The asymptotical behavior of the α_{\pm} and $\tilde{\alpha}_-$ functions with the use of (24) – (28) is presented on Fig.5 for the case of a sufficiently small value of x_0 . As one can see, this

behavior is quite similar to the behavior for the case of $x_0 = 10^{-3}$, compare with Fig.2 and Fig.4. It should be noted that the $f_{\pm}(x_0, x)$ functions depend strongly on x_0 .

3 Total vacuum energy and the Casimir force

The total vacuum energy which is induced outside the magnetic flux tube of finite radius is

$$E \equiv \int_0^{2\pi} d\varphi \int_{r_0}^{\infty} \varepsilon_{ren} r dr = 2\pi m \left[\int_{x_0}^{\infty} \frac{\alpha_+(x_0, x)}{x^2} dx - \left(\xi - \frac{1}{4} \right) \int_{x_0}^{\infty} \frac{\tilde{\alpha}_-(x_0, x)}{x^2} dx \right]. \quad (29)$$

In view of the relation

$$\int_{x_0}^{\infty} \frac{\tilde{\alpha}_-(x_0, x)}{x^2} dx = \left[\frac{\alpha_-(x_0, x)}{x} - \frac{\partial \alpha_-(x_0, x)}{\partial x} \right] \Big|_{x=x_0} \quad (30)$$

which follows from (13), and relations (24) and (27), we conclude that the vacuum energy is independent of the coupling to the space-time curvature scalar (ξ):

$$E = 2\pi m \int_{x_0}^{\infty} \frac{\alpha_+(x_0, x)}{x^2} dx. \quad (31)$$

This is in contrast to the case of the singular magnetic thread, when the total induced vacuum energy is divergent and ξ -dependent (see [14]):

$$E^{sing} \equiv \int_0^{2\pi} d\varphi \int_0^{\infty} \varepsilon_{ren}^{sing} r dr \sim 4m \left(\xi - \frac{1}{12} \right) \int_0^{\infty} \frac{dx}{x^2}. \quad (32)$$

It is curious that the vacuum energy in this case is finite at $\xi = 1/12$, being equal to

$$E^{sing}|_{\xi=1/12} = \frac{2m}{3\pi} \int_0^{\infty} \left\{ \frac{\pi}{2} - \left(2x + \frac{1}{2x} \right) K_0(2x) - K_1(2x) - \pi x [K_0(2x)L_1(2x) + K_1(2x)L_0(2x)] \right\} x dx = -0.01989 \times 2\pi m. \quad (33)$$

Although vacuum energy E (31) is finite, its absolute value grows infinitely as x_0 tends to zero (see (24) and (26)):

$$E|_{x_0 \rightarrow 0} \sim -\frac{m}{8\pi x_0}, \quad (34)$$

which is in accordance with the divergence of the vacuum energy in the case of the singular magnetic thread. To be more precise, relation (30) fails to yield zero in the case $x_0 = 0$, and, therefore, the divergence of the vacuum energy in the latter case becomes ξ -dependent.

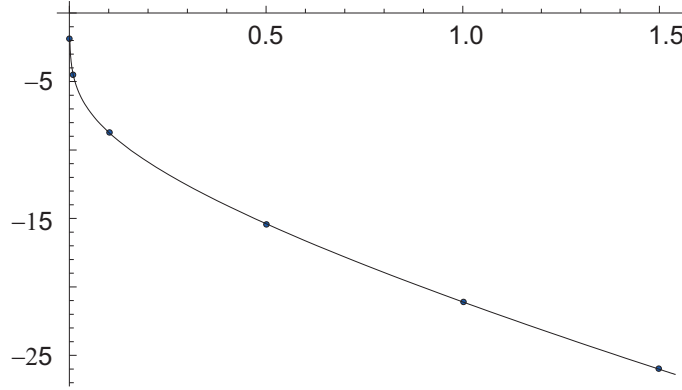


Figure 6: The total vacuum energy as a function of the tube radius in the range $10^{-3} < x_0 < 3/2$. The variable x_0 is along the abscissa axis, the value of $\ln \frac{E}{2\pi m}$ is along the ordinate axis. Solid line interpolates the dots that have been calculated.

We present the values of vacuum energy E (31) for several values of the tube radius in the Table:

x_0	$3/2$	1	$1/2$	10^{-1}	10^{-2}	10^{-3}
$E/(2\pi m)$	$5.013 \cdot 10^{-12}$	$6.944 \cdot 10^{-10}$	$2.068 \cdot 10^{-7}$	$1.65 \cdot 10^{-4}$	0.0106	0.1486

These results are also given on Fig.6 in logarithmic scale, where the dots corresponding to the data in the Table are joined with the help of the interpolation function,

$$\eta(x_0) = \ln \frac{E}{2\pi m}, \quad (35)$$

which can be taken in the form

$$\eta(x_0) = a_0 + \sum_i a_i x_0^{b_i} - \left(1 + \sum_j c_j x_0^{d_j} \right) \ln x_0. \quad (36)$$

where b_i and d_j are the positive adjustable constants.

To change the radius of the magnetic flux tube one has to apply a work that is equal to the change of the total vacuum energy which is induced outside the tube. In the case of the infinitely small change of the radius one has

$$\Delta E = 2\pi p r_0 \Delta r_0, \quad (37)$$

where p can be interpreted as the vacuum pressure which acts from the outside to the inside of the tube

$$p(x_0) = \frac{1}{2\pi r_0} \frac{dE}{dr_0} = m^3 \frac{e^{\eta(x_0)}}{x_0} \frac{d\eta(x_0)}{dx_0}. \quad (38)$$

This results in the Casimir force acting from the inside to the outside of the tube

$$F(x_0) = -2\pi r_0 p(x_0) = -2\pi m^2 e^{\eta(x_0)} \frac{d\eta(x_0)}{dx_0}. \quad (39)$$

The behavior of the Casimir force is presented on Fig.7.

As one can see, the Casimir force tends to increase the radius of the tube and to minimize the induced vacuum energy of the quantized scalar field. Certainly, our conclusion is obtained under the assumption that the magnetic flux inside the impenetrable tube remains invariable with the variation of the tube radius.

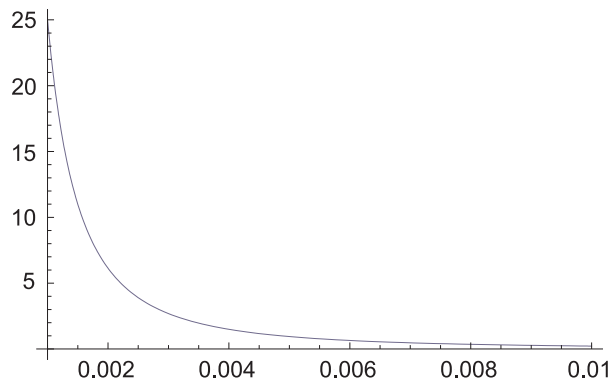


Figure 7: The Casimir force as a function of the tube radius. The variable x_0 is along the abscissa axis, the value of the dimensionless Casimir force $\frac{F(x_0)}{2\pi m^2}$ is along the ordinate axis.

4 Summary

In the present paper we consider the vacuum polarization effects which are induced in scalar matter by imposing a perfectly reflecting (Dirichlet) boundary condition at the edge of an impenetrable finite-radius tube which carries magnetic flux lines inside itself. Restricting ourselves to a plane which is orthogonal to the tube, we define the induced vacuum energy density, see (7) and (8), and analyze numerically its behavior as a function of the distance from the tube for the magnetic flux equal to half of the London flux quantum ($\Phi = \pi e^{-1}$), the tube radius equal to $r_0 = 10^{-3}m^{-1}$ and different values of the coupling to the space-time curvature scalar (ξ), see Fig.3. The emergence of the energy density, as well as of other components of the energy-momentum tensor, in the vacuum can lead to various semiclassical gravitational effects which were estimated under the neglect of the tube radius in [15].

The present paper summarizes and extends our previous study in [11, 12], and this allows us to draw conclusions about the behavior of the total induced vacuum energy, i.e. the density integrated over the whole plane, and the Casimir force as functions of the tube radius. We find that the total induced vacuum energy is finite and independent of ξ , as long as the tube radius is taken into account. Although the values of the total induced vacuum energy are negligible for $r_0 \sim m^{-1}$ (see also [12]) being of order $10^{-10} \times 2\pi m$, they are of order $10^{-1} \times 2\pi m$ for $r_0 \sim 10^{-3}m^{-1}$, see the Table and Fig.6. The induced vacuum energy gives rise to the Casimir force which is directed from the inside to the outside of the tube. The force acts at the increase of the tube radius and the decrease of the induced vacuum energy under the steady magnetic flux. The force takes considerable values at small values of the tube radius and actually disappears otherwise: it is, e.g., $10 \times 2\pi m^2$ at $r_0 \sim 10^{-3}m^{-1}$ and $10^{-4} \times 2\pi m^2$ at $r_0 \sim 10^{-1}m^{-1}$. The behavior of the force as a function of the tube radius is illustrated by Fig.7.

It should be noted that in our case the Casimir force is caused by boundary conditions imposed at the boundary enclosing a magnetic flux. The force is periodic in the flux value with a period equal to the London flux quantum, attaining its maximal value at $\Phi = (2n + 1)\pi e^{-1}$ and vanishing at $\Phi = 2n\pi e^{-1}$ ($n \in \mathbb{Z}$).

Acknowledgments

The work was partially supported by special program "Microscopic and phenomenological models of fundamental physical processes in micro- and macroworld" of the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine.

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